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Modular Forms and Vertex Operator Algebras

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Thesis

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science at Virginia Commonwealth University.

by

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Abstract

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In this thesis we present the connection between vertex operator algebras and modular forms which lies at the heart of Borcherds' proof of the Monstrous Moonshine conjecture. In order to do so we introduce modular forms, vertex algebras, vertex operator algebras and their partition functions. Each notion is illustrated with examples.

Introduction

The definition of vertex operator algebra (VOA) was introduced in 1992 by Richard Borcherds [1] to resolve the Conway-Norton conjecture which predicted an unexpected connection between the largest finite simple group (the "Monster" group) and the Fourier expansion of the j-invariant,

$$j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots,$$

a modular function which parameterizes elliptic curves up to isomorphism. Because of this connection's mysterious nature, this relationship became known as Monstrous Moonshine. Borcherds would later win the Fields Medal for his work in resolving the Moonshine conjecture.

A VOA is a vector space V together with a collection of operators acting on it satisfying suitable axioms. This includes a chosen operator L_0 such that there is a decomposition $V = \bigoplus_{n\geq 0} V_n$ into eigenspaces of L_0. To this VOA one can attach a function on the upper half plane of the form

$$Z(q) = q^{-c/24} \sum_{n\in Z} \dim V_n q^n$$

called the partition function. In many examples, the partition function of a VOA happens to be a modular form. However, this process is not straightforward and does not work for some VOAs. There is much active research being done on this relationship. Borcherds defined a VOA where the V_n are constructed from representations of the Monster group, and also match exactly the coefficients in the j-invariant. The notion of a VOA arises quite naturally from physics, specifically, conformal field theory. In this case, of which Borcherds' construction is an example, the modularity found from the partition function of a VOA is not surprising; rather it is a property expected given the symmetries of these physical theories.

Because the history of VOAs spans many diverse areas, a full treatment is not possible, and so we endeavor to present only the minimal background necessary to understand the final statement given in this thesis. We cannot include all the remarkable connections to physics and will focus only on the mathematical constructions. Hence, we will introduce modular forms, theta functions, vertex algebras (with examples), VOAs, and their partition functions.

Modular forms

2.1 Modular group

Let $\mathbb{H}$ denote the upper half of the complex plane, that is, the set of complex numbers z with imaginary part $\mathrm{Im}(z) > 0$. Let $\mathrm{SL}_2(\mathbb{R})$ be the group of 2×2 real matrices having determinant 1. Now define $\widetilde{\mathbb{C}} = \mathbb{C} \cup \{ \infty \}$ and make $\mathrm{SL}_2(\mathbb{R})$ act on $\widetilde{\mathbb{C}} \setminus \mathbb{R}$ in the following way: if $z \in \widetilde{\mathbb{C}} \setminus \mathbb{R}$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$, we put

$$gz = \frac{az+b}{cz+d}.$$

This action is also known as a *Möbius transformation*. Since

$$\mathrm{Im}(gz) = \frac{\mathrm{Im}(z)}{|cz+d|^2},$$

it follows that $\mathbb{H}$ is stable under the action of $\mathrm{SL}_2(\mathbb{R})$. Also note that the element $-1 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ acts trivially on $\mathbb{H}$. Thus we may consider the group $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{ \pm 1 \}$, which can be shown to be the group of all analytic automorphisms of $\mathbb{H}$. Let $\mathrm{SL}_2(\mathbb{Z})$ be the subgroup of $\mathrm{SL}_2(\mathbb{R})$ consisting of only the matrices with coefficients in $\mathbb{Z}$; this is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$.

DEFINITION 2.1. The group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{ \pm 1 \}$ is called the *modular group*.

Note that $\mathrm{PSL}_2(\mathbb{Z})$ is the image of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{PSL}_2(\mathbb{R})$. If $g \in \mathrm{SL}_2(\mathbb{Z})$, we use the same symbol to denote its image in the modular group.

2.1.1 The fundamental domain of the modular group

This section follows the work done in Serre [4]. Let $S, T \in \mathrm{PSL}_2(\mathbb{Z})$ with

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the following is true:

$$Sz = -1/z, \qquad Tz = z+1, \qquad S^2 = 1, \qquad (ST)^3 = 1.$$

Now let

$$D = \{\, z \in \mathbb{H} \mid |z| \geq 1 \text{ and } |\mathrm{Re}(z)| \leq 1/2 \,\}.$$

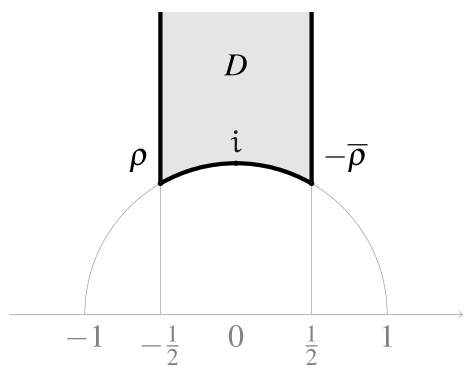

Figure 2.1: The fundamental domain D of $\mathrm{PSL}_2(\mathbb{Z})$.

Using the following theorem, we show that D is the *fundamental domain* for the action of G on $\mathbb{H}$.

THEOREM 2.2. 1. For every $z \in \mathbb{H}$, there exists $g \in \mathrm{PSL}_2(\mathbb{Z})$ such that $gz \in D$.

2. Let z, z' be distinct points in D that are congruent modulo $\mathrm{PSL}_2(\mathbb{Z})$. Then $\mathrm{Re}(z) = \pm 1/2$ and $z = z' \pm 1$, or $|z| = 1$ and $z' = -1/z$.

3. Let $z \in D$ and let $I(z) = \{ g \in \mathrm{PSL}_2(\mathbb{Z}) \mid gz = z \}$, that is, the stabilizer of z in $\mathrm{PSL}_2(\mathbb{Z})$. We have $I(z) = \{ 1 \}$ except in the following cases:

   - $z = \mathrm{i}$, in which case $I(z)$ is the group of order 2 generated by S;
   - $z = \rho = \mathrm{e}^{2\pi\mathrm{i}/3}$, in which case $I(z)$ is the group of order 3 generated by ST;
   - $z = -\overline{\rho} = \mathrm{e}^{\pi\mathrm{i}/3}$, in which case $I(z)$ is the group of order 3 generated by TS.

The first two assertions of the theorem imply the following corollary.

COROLLARY 2.3. The canonical map $D \to \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ is surjective and its restriction to the interior of D is injective.

THEOREM 2.4. $\mathrm{PSL}_2(\mathbb{Z})$ is generated by S and T.

2.2 Modular functions

DEFINITION 2.5. Let k be an integer. A function $f : \mathbb{H} \to \mathbb{C}$ is *weakly modular of weight* $2k$ if f is meromorphic on $\mathbb{H}$ and verifies the relation

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.1}$$

Let g be the image in $\mathrm{PSL}_2(\mathbb{Z})$ of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We have $\mathrm{d}(gz)/\mathrm{d}z = (cz+d)^{-2}$. Then equation (2.1) can be written:

$$\frac{f(gz)}{f(z)} = \left(\frac{\mathrm{d}(gz)}{\mathrm{d}z}\right)^{-k}$$

or

$$f(gz)\,\mathrm{d}(gz)^k = f(z)(\mathrm{d}z)^k.$$

We can interpret this as meaning that the "differential form of weight k" $f(z)\,\mathrm{d}z^k$ is *invariant* under $\mathrm{PSL}_2(\mathbb{Z})$. Since $\mathrm{PSL}_2(\mathbb{Z})$ is generated by the elements S and T (from Theorem 2.4), it suffices to check the invariance by S and by T. This gives the following property of weakly modular functions:

COROLLARY 2.6. Let f be meromorphic on $\mathbb{H}$. The function f is a weakly modular function of weight $2k$ if and only if it satisfies the two relations:

$$f(z+1) = f(z)$$

$$f(-1/z) = z^{2k} f(z).$$

If the first relation is verified, we can then write f as a function of $q = \mathrm{e}^{2\pi \mathrm{i} z}$, which we will denote $\tilde{f}$. Note that $\tilde{f}$ is meromorphic in the disk $|q| < 1$ with the origin removed.

DEFINITION 2.7. If $\tilde{f}$ may be extended to a meromorphic (holomorphic) function at the origin, we say that f is *meromorphic (holomorphic) at infinity*.

This means that $\tilde{f}$ admits a Laurent expansion in a neighborhood around the origin

$$\tilde{f}(q) = \sum_{n \in \mathbb{Z}} a_n q^n,$$

where the a_n is zero for small enough n.

DEFINITION 2.8. A *modular function* is a weakly modular function that is holomorphic at infinity. If f is holomorphic at infinity, we set $f(\infty) = \tilde{f}(0)$ and call that the *value* of f at infinity. A modular function which is holomorphic everywhere (including infinity) is called a *modular form*. If such a function is zero at infinity, it is called a *cusp form*. A modular form of weight $2k$ is given by a series

$$f(z) = \sum_{n=0}^{\infty} a_n q^n = \sum_{n=0}^{\infty} a_n \mathrm{e}^{2\pi \mathrm{i} n z}$$

which converges for $|q| < 1$ (that is, for $\mathrm{Im}(z) > 0$), and which verifies the identity

$$f(-1/z) = z^{2k} f(z).$$

Note that if the coefficient a_0 is 0, then f is a cusp form.

Note that if f and f' are modular forms of weight $2k$ and $2k'$ respectively, their product ff' is also a modular form of weight $2k+2k'$. More generally, we say $f(z)$ is a modular form of weight $\pm p/q$, with $p,q \in \mathbb{Z}_+$, if $f(z+1) = f(z), f(-1/z) = z^{\pm 2p/q} f(z)$, and $(f(z))^{\pm 2q}$ is a modular form of weight $2p$.

2.2.1 Eisenstein series

The Eisenstein series serves as our first example and will be useful in discussing the space of modular forms. This section combines the discussion of the Eisenstein series in both Serre [4] and Stein [5].

DEFINITION 2.9. Let Γ be a lattice of $\mathbb{C}$, and let $k > 1$ be an integer. The *Eisenstein series of weight* $2k$ is a function on $\mathbb{H}$ defined as

$$G_k(z) = \sum_{(n,m)\neq(0,0)} \frac{1}{(mz+n)^{2k}}.$$

THEOREM 2.10. Eisenstein series have the following properties:

1. The series $G_k(z)$ converges if $k > 1$, and is holomorphic in $\mathbb{H}$.
2. $G_k(z+1) = G_k(z)$ and $G_k(z) = z^{-k} G_k(-1/z)$.
3. $G_k(z)$ is a modular form of weight $2k$.
4. $G_k(\infty) = 2\zeta(2k)$ where ζ is the Riemann zeta function.

We first state the following lemma and its proof [5] in order to prove the convergence of $G_k(z)$:

LEMMA 2.11. Let $\Gamma = \{\, n + m\tau \mid n, m \in \mathbb{Z} \,\}$, and $\Gamma' = \Gamma \setminus \{\, (0,0) \,\}$, that is, a lattice with the origin removed. The two series

$$\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} \qquad \text{and} \qquad \sum_{n+m\tau\in\Gamma'} \frac{1}{|n+m\tau|^r}$$

converge if $r > 2$.

Proof. The question of whether a double series converges absolutely is independent of the order of summation; in this case we first sum in m and then in n. For the first series, the usual integral comparison can be applied. For each $n \neq 0$,

$$\begin{aligned}
\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} &= \frac{1}{|n|^r} + 2\sum_{m\geq 1} \frac{1}{(|n|+|m|)^r} \\
&= \frac{1}{|n|^r} + 2\sum_{k\geq |n|+1} \frac{1}{k^r} \\
&\leq \frac{1}{|n|^r} + 2\int_{|n|}^{\infty} \frac{\mathrm{d}x}{x^r} \\
&\leq \frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}},
\end{aligned}$$

where C is the constant of integration. Therefore, $r > 2$ implies

$$\begin{aligned}
\sum_{(n,m)\neq(0,0)} \frac{1}{(|n|+|m|)^r} &= \sum_{|m|^r} + \sum_{|n|\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(|n|+|m|)^r} \\
&\leq \sum_{|m|\neq 0} \frac{1}{|m|^r} + \sum_{|n|\neq 0} \left(\frac{1}{|n|^r} + C\frac{1}{|n|^{r-1}} \right) \\
&< \infty.
\end{aligned}$$

To prove that the second series also converges, it suffices to show that there is a constant c such that $|n|+|m| \leq c\,|n+m\tau|$ for all $n,m \in \mathbb{Z}$.

We use the notation $x \lesssim y$ if there exists a positive constant a such that $x \leq ay$. We also write $x \approx y$ if both $x \lesssim y$ and $y \lesssim x$ hold. Note that for any two positive numbers A and B, we have

$$(A^2+B^2)^{1/2} \approx A+B.$$

On one hand, $A \leq (A^2+B^2)^{1/2}$ and $B \leq (A^2+B^2)^{1/2}$, so that $A+B \leq 2(A^2+B^2)^{1/2}$. On the other hand, it suffices to square both sides to see that $(A^2+B^2)^{1/2} \leq A+B$. The proof that the second series converges is now a consequence of the observation that

$$|n|+|m| \approx |n+m\tau| \quad \text{whenever } \tau \in \mathbb{H}.$$

If we write $\tau = s+\imath t$, with $s,t \in \mathbb{R}$ and $t > 0$, then

$$|n+m\tau| = [(n+ms)^2+(mt)^2]^{1/2} \approx |n+ms|+|mt| \approx |n+ms|+|m|$$

by the previous observation. Then,

$$|n+ms|+|m| \approx |n|+|m|,$$

by considering the two cases when $|n| \leq 2\,|m|\,|s|$ and $|n| \geq 2\,|m|\,|s|$.

This proof shows that when $r > 2$ the series $\sum |n+m\tau|^{-r}$ converges uniformly in every half-plane $\operatorname{Im}(\tau) \geq \delta > 0$. In contrast, when $r = 2$ this series fails to converge. □

Now we can prove Theorem 2.10.

Proof. From the above lemma, the series $G_k(z)$ converges absolutely and uniformly in every half-plane $\operatorname{Im}(z) \geq \delta > 0$, whenever $k > 1$; hence $G_k(z)$ is holomorphic in $\mathbb{H}$, which gives us (1). Clearly $G_k(z)$ is periodic and has period 1 since $n+m(z+1) = n+m+mz$, and that we can rearrange the sum by replacing $n+m$ by n. Also, we have

$$(n+m(-1/z))^k = z^{-k}(nz-m)^k,$$

and again we can rearrange the sum, this time replacing $(-m,n)$ by (n,m), and so (2) follows. Property (3) follows directly from (1) and (2). To see property (4), observe that

$$\lim_{\mathrm{Im}(z)\to\infty} G_k(z) = \sum_{n\neq 0} \frac{1}{n^{2k}} = 2\sum_{n=1}^{\infty} \frac{1}{n^{2k}} = 2\zeta(2k).$$

□

The Eisenstein series of lowest weights are G_2 and G_3, which are of weight 4 and 6 respectively. Because of their significance to the theory of elliptic curves (which is beyond the scope of this thesis), we define

$$g_2 = 60G_2, \qquad g_3 = 140G_3.$$

Then have we have $g_2(\infty) = 120\zeta(4)$ and $g_3(\infty) = 280\zeta(6)$. Since $\zeta(4) = \pi^4/90$ and $\zeta(6) = \pi^6/945$, we can write

$$g_2(\infty) = \frac{4}{3}\pi^4, \quad g_3(\infty) = \frac{8}{27}\pi^6.$$

In particular, the *modular discriminant*

$$\Delta = g_2^3 - 27g_3^2$$

is a cusp form of weight 12.

2.3 Space of modular forms

DEFINITION 2.12. Let f be a meromorphic function on $\mathbb{H}$ that is not identically zero, and let p be a point in $\mathbb{H}$. Then the largest integer n such that $f(z)/(z-p)^n$ is holomorphic and non-zero at p is called the *order of f at p* and is denoted $\nu_p(f)$.

REMARK 2.13. The order of f at p is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$, that is, $\nu_p(f) = \nu_{g(p)}(f)$ for $g \in \mathrm{PSL}_2(\mathbb{Z})$.

Proof. Suppose that $\nu_p(f) = n$. If we take the Laurent expansion of $f(z)$ at p

$$f(z) = \frac{a_{-n}}{(z-p)^n} + \frac{a_{-n+1}}{(z-p)^{n-1}} + \cdots + a_0 + a_1(z-p) + \cdots$$

and apply the transformation $z \mapsto z+1$, we have

$$\begin{aligned} f(z+1) &= \frac{a_{-n}}{((z+1)-(p+1))^n} + \frac{a_{-n+1}}{((z+1)-(p+1))^{n-1}} + \cdots \\ &= \frac{a_{-n}}{(z-p)^n} + \frac{a_{-n+1}}{(z-p)^{n-1}} + \cdots \end{aligned}$$

Thus $f(z)$ has a pole or zero of order n if and only if $f(z+1)$ has one as well. Furthermore, since $f(z)$ is a modular function, we can take the identity

$$f(z) = (cz+d)^{-2k} f\left(\frac{az+b}{cz+d}\right),$$

and substitute in the representation of this transformation, the matrix T from above, to see that $f(z) = f(z+1)$.

Similarly, we apply the transformation $z \mapsto -1/z$ to the Laurent expansion of $f(z)$ to get

$$f(-1/z) = \frac{a_{-n}}{\left(\frac{-1}{z} + \frac{1}{p}\right)^n} + \cdots = \frac{a_{-n}}{\left(\frac{z-p}{pz}\right)^n} + \cdots = \frac{a_{-n} p^n z^n}{(z-p)^n} + \cdots$$

Since $p^n z^n$ is just a positive number, it does not affect our result. We then use the generating matrix S from above in the identity to get

$$f(z) = z^{-2k} f(-1/z).$$

Therefore, the order $v_p(f)$ is invariant under the action of $\mathrm{PSL}_2(\mathbb{Z})$. $\square$

We can also define $v_\infty(f)$ as the order for $q = 0$ of the function $\tilde{f}(q)$ associated to f (cf. Section 2.2).

Denote by e_p the order of the stabilizer of p. If p is congruent modulo $\mathrm{PSL}_2(\mathbb{Z})$ to i then $e_p = 2$. If instead p is congruent modulo $\mathrm{PSL}_2(\mathbb{Z})$ to $\rho = \mathrm{e}^{2\pi\mathrm{i}/3}$, then $e_p = 3$. Otherwise, $e_p = 1$.

PROPOSITION 2.14. Let f be a modular function of weight $2k$ that is not identically zero. Then,

$$v_\infty(f) + \frac{1}{2} v_{\mathrm{i}}(f) + \frac{1}{3} v_\rho(f) + \sum_{p \in \mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})}^{*} v_p(f) = \frac{k}{6}, \tag{2.2}$$

where the symbol $\sum^*$ means a summation over points in $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ distinct from the equivalency classes of i and ρ [4].

Proof. We shall integrate $\frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}f}{f}$ along the boundary of the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$.

Suppose that f has no poles or zeroes on the boundary of D except possibly at i, ρ, and $-\overline{\rho}$. (Any other poles or zeroes along these half-lines can be easily dealt with by slight

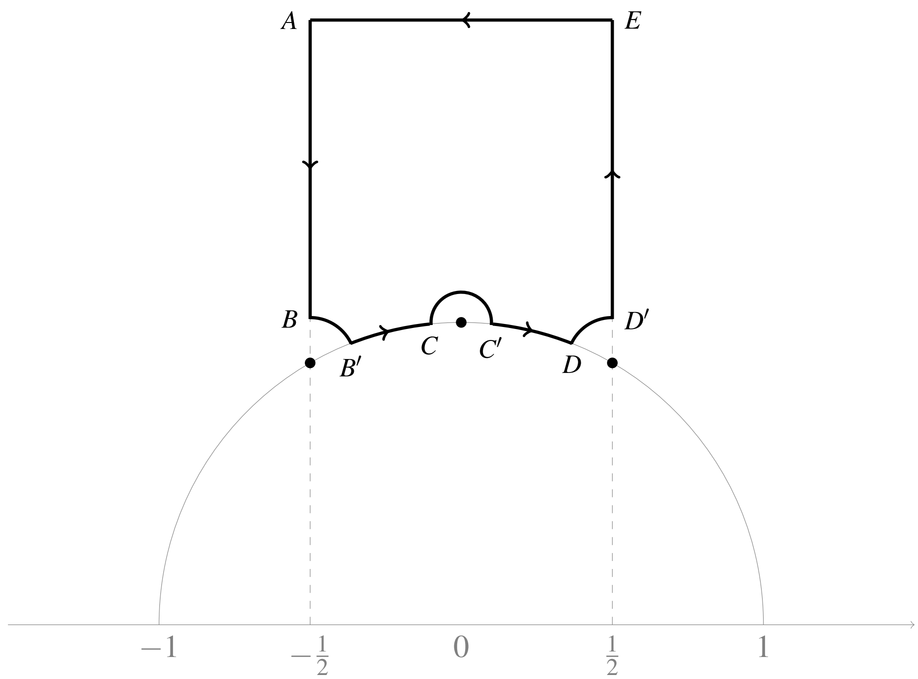

Figure 2.2: The contour $\mathscr{C}$.

modification of the contour and using the $\mathrm{PSL}_2(\mathbb{Z})$ symmetry.) Then there is a contour $\mathscr{C}$ (see Figure 2.2) whose interior contains a representative of each pole or zero of f not congruent to i or ρ. By the residue theorem, we have

$$\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}}\frac{\mathrm{d}f}{f}=\sum_{p\in\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})}^{*}\nu_p(f).$$

The top segment EA of the contour may be transformed by the change of variables $q=\mathrm{e}^{2\pi\mathrm{i}z}$ into a circle ω centered at 0 to get

$$\frac{1}{2\pi\mathrm{i}}\int_E^A\frac{\mathrm{d}f}{f}=\frac{1}{2\pi\mathrm{i}}\int_\omega\frac{\mathrm{d}f}{f}=-\nu_\infty(f).$$

The integral of $\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}f}{f}$ on the circle which contains the arc BB', oriented negatively, has the value $-\nu_\rho(f)$. As the radius r_1 of this circle goes to 0, the angle between B and B' goes to

$2\pi/6$. Hence,

$$\lim_{r_1\to 0}\frac{1}{2\pi \mathrm{i}}\int_B^{B'}\frac{\mathrm{d}f}{f}=-\frac{1}{6}v_\rho(f).$$

Similarly, if we let the radii r_2 and r_3 of the arcs CC' and DD', respectively, go to 0, we have

$$\lim_{r_2\to 0}\frac{1}{2\pi \mathrm{i}}\int_C^{C'}\frac{\mathrm{d}f}{f}=-\frac{1}{2}v_\mathrm{i}(f),$$
$$\lim_{r_3\to 0}\frac{1}{2\pi \mathrm{i}}\int_D^{D'}\frac{\mathrm{d}f}{f}=-\frac{1}{6}v_\rho(f).$$

T transforms the arc AB into the arc ED', and since $f(Tz)=f(z)$, we have

$$\frac{1}{2\pi \mathrm{i}}\int_A^{B}\frac{\mathrm{d}f}{f}+\frac{1}{2\pi \mathrm{i}}\int_{D'}^{E}\frac{\mathrm{d}f}{f}=0.$$

S transforms the arc $B'C$ into the arc DC', and since $f(Sz)=z^{2k}f(z)$, we have

$$\frac{\mathrm{d}f(Sz)}{f(Sz)}=2k\frac{\mathrm{d}z}{z}+\frac{\mathrm{d}f(z)}{f(z)},$$

and thus

$$\begin{aligned}\frac{1}{2\pi \mathrm{i}}\int_{B'}^{C}\frac{\mathrm{d}f}{f}+\frac{1}{2\pi \mathrm{i}}\int_{C'}^{D}\frac{\mathrm{d}f}{f}&=\frac{1}{2\pi \mathrm{i}}\int_{B'}^{C}\left(\frac{\mathrm{d}f(z)}{f(z)}-\frac{\mathrm{d}f(Sz)}{f(Sz)}\right)\\&=\frac{1}{2\pi \mathrm{i}}\int_{B'}^{C}\left(-2k\frac{\mathrm{d}z}{z}\right).\end{aligned}$$

When we let the radii of the arcs BB', CC', and DD' go to 0, we have

$$\lim_{r_1,r_2,r_3\to 0}\frac{1}{2\pi \mathrm{i}}\int_{B'}^{C}\left(-2k\frac{\mathrm{d}z}{z}\right)=-2k\left(-\frac{1}{12}\right)=\frac{k}{6}.$$

We can now set the two different expressions for $\frac{1}{2\pi \mathrm{i}}\int_{\mathscr{C}}\frac{\mathrm{d}f}{f}$ equal, and again take the limit to find the desired formula.

$\square$

For an integer k, we denote the $\mathbb{C}$-vector space of modular forms of weight $2k$ by M_k (and similarly the cusp forms of weight $2k$ by M_k^0). By definition, M_k^0 is the kernel of the linear form $f \mapsto f(\infty)$ on M_k. Thus we have $\dim M_k/M_k^0 \leq 1$. For $k \geq 2$, the Eisenstein series G_k (see Section 2.2.1) is an element of M_k such that $G_k(\infty) \neq 0$, therefore we have that

$$M_k = M_k^0 \oplus \mathbb{C}G_k,$$

where $\mathbb{C}G_k$ is the complex vector space spanned by G_k.

THEOREM 2.15. The following statements are true:

1. If $k < 0$ or $k = 1$, $M_k = 0$.

2. For $k = 0, 2, 3, 4, 5$, respectively, M_k is a vector space of dimension 1 with basis $1, G_2, G_3, G_4, G_5$, respectively. Furthermore $M_k^0 = 0$.

3. Multiplication by Δ defines an isomorphism from M_{k-6} onto M_k^0. (Recall from above that $\Delta = g_2^3 - 27g_3^2$.)

Proof. We give a proof of the third statement of Theorem 2.15. Consider (2.2) above with $f = G_k, k = 2$. Write $2/6$ in the form $n + n'/2 + n''/3$ where $n, n', n'' \geq 0$ only when $n = 0, n' = 0, n'' = 1$. Hence $\nu_\rho(G_2) = 1$ and $\nu_\rho(G_2) = 0$ for $p \neq \rho$ (modulo $\mathrm{PSL}_2(\mathbb{Z})$). Apply a similar argument to G_3 and thus $\nu_{\mathrm{i}}(G_3) = 1$ and all the others $\nu_p(G_3) = 0$. This shows that Δ is not zero at i, and hence Δ is not identically zero. Since Δ is of weight 12 and $\nu_\infty(\Delta) \geq 1$, (2.2) implies that $\nu_p(\Delta) = 0$ for $p \neq 0$ and $\nu_\infty(\Delta) = 1$. That is, Δ does not vanish on $\mathbb{H}$ and has a simple zero at infinity. If $f \in M_k^0$, and we set $g = f/\Delta$, then it is easy to show that g is of weight $2k-12$. The formula

$$v_p(g) = v_p(f) - v_p(\Delta) = \begin{cases} v_p(f) & p \neq \infty, \\ v_p(f) - 1 & p = \infty \end{cases}$$

implies $v_p(g) \geq 0$ for all p, and so $g \in M_{k-6}$. □

COROLLARY 2.16. We have

$$\dim M_k = \begin{cases} [k/6], & k \equiv 1 \pmod 6, k \geq 0 \\ [k/6]+1, & k \not\equiv 1 \pmod 6, k \geq 0. \end{cases}$$

(Here $[x]$ denotes the largest integer n such that $n \leq x$.)

COROLLARY 2.17. The space M_k has for a basis the family of monomials $G_2^{\alpha} G_3^{\beta}$ where α, β are non-negative integers with $2\alpha + 3\beta = k$.

For example, $G_4 = \frac{9}{2\pi^2} G_2^2 G_3^0$, since $2(2)+3(0) = 4$. The coefficient $9/2\pi^2$ results from the property stated in Theorem 2.10, where we have computed that $\zeta(8)/\zeta(4)\zeta(6) = 9/\pi^2$. Because this is the only way to write 4 as a linear combination of 2 and 3 with non-negative coefficients, this is the only possible way to write a basis of M_4 as such a family of monomials. Similarly, $G_5 = \frac{5}{11} G_2^1 G_3^1$ since $2(1)+3(1) = 5$.

2.4 Poisson summation formula

DEFINITION 2.18. Let V be a $\mathbb{R}$-vector space of finite dimension n with an invariant measure μ. Denote the dual of V by V^*. Let f be a rapidly decreasing smooth function on V. Then the *Fourier transform* $\widehat{f}$ *of* f is defined as

$$\widehat{f}(y) = \int_V e^{-2\pi i\langle x,y\rangle} f(x)\mu(x).$$

This is a rapidly decreasing smooth function on V^*. The Poisson summation formula [5] gives us a relationship between a function and its Fourier transform. For each $a > 0$, we denote by $\mathscr{F}_a$ the class of all functions f that satisfy the following two conditions:

1. The function f is holomorphic in the horizontal strip $S_a = \{ z \in \mathbb{C} \mid |\mathrm{Im}(z)| < a \}$.

2. There exists a constant $A > 0$ such that

$$|f(x+iy)| \leq \frac{A}{1+x^2} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| < a.$$

In other words, $\mathscr{F}_a$ consists of the holomorphic functions on S_a that are of moderate decay on each horizontal line $\mathrm{Im}(z) = y$, uniformly in $-a < y < a$. We denote by $\mathscr{F}$ the class of all functions that belong to $\mathscr{F}_a$ for some a.

THEOREM 2.19 (Poisson summation formula [5]). If $f \in \mathscr{F}$, then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n).$$

Proof. Say $f \in \mathscr{F}_a$ and choose some b satisfying $0 < b < a$. The function $1/(e^{2\pi iz} - 1)$ has simple poles with residue $1/(2\pi i)$ at the integers. Thus $f(z)/(e^{2\pi iz} - 1)$ has simple poles at the integers n, with residues $f(n)/2\pi i$. Therefore we may apply the residue formula to the contour γN where N is an integer, a rectangle centered at the origin of width $2N$ and height 1. This yields

$$\sum_{|n|\leq N} f(n) = \int_{\gamma N} \frac{f(z)}{e^{2\pi iz} - 1}\,dz.$$

Letting N go to infinity and recalling that f has moderate decrease, we see that the sum converges to $\sum_{n\in\mathbb{Z}} f(n)$, and also that the integral over the vertical segments of the rectangle cancel each other out. Therefore, in the limit we have

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{\mathrm{e}^{2\pi \mathrm{i} z}-1}\,\mathrm{d}z - \int_{L_2} \frac{f(z)}{\mathrm{e}^{2\pi \mathrm{i} z}-1}\,\mathrm{d}z, \tag{2.3}$$

where L_1 and L_2 are the real line shifted down and up by b, respectively.

Now we use the fact that if $|w| > 1$, then

$$\frac{1}{w-1} = w^{-1}\sum_{n=0}^{\infty} w^{-n}$$

to see that on L_1 (where $\left|\mathrm{e}^{2\pi \mathrm{i} z}\right| > 1$) we have

$$\frac{1}{\mathrm{e}^{2\pi \mathrm{i} z}-1} = \mathrm{e}^{-2\pi \mathrm{i} z}\sum_{n=0}^{\infty} \mathrm{e}^{-2\pi \mathrm{i} n z}.$$

Also if $|w| < 1$, then

$$\frac{1}{w-1} = -\sum_{n=0}^{\infty} w^{n}$$

so that on L_2

$$\frac{1}{\mathrm{e}^{2\pi \mathrm{i} z}-1} = -\sum_{n=0}^{\infty} \mathrm{e}^{2\pi \mathrm{i} n z}.$$

Substituting these observations into (2.3), we find that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f(n) &= \int_{L_1} f(z)\left(e^{-2\pi i z}\sum_{n=0}^{\infty} e^{-2\pi i n z}\right)dz + \int_{L_2} f(z)\left(\sum_{n=0}^{\infty} e^{2\pi i n z}\right)dz \\
&= \sum_{n=0}^{\infty}\int_{L_1} f(z)e^{-2\pi i(n+1)z}\,dz + \sum_{n=0}^{\infty}\int_{L_2} f(z)e^{2\pi i n z}\,dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi i(n+1)x}\,dx + \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(x)e^{2\pi i n x}\,dz \\
&= \sum_{n=0}^{\infty}\widehat{f}(n+1) + \sum_{n=0}^{\infty}\widehat{f}(-n) \\
&= \sum_{n\in\mathbb{Z}}\widehat{f}(n),
\end{aligned}$$

where we have shifted L_1 and L_2 back to the real line. $\square$

We also state the more general version of the Poisson summation formula for lattices [4], which will become useful later when defining the theta function for a lattice.

Let Γ be a lattice in V^*. Then denote by Γ^* the lattice in V^* that is dual to Γ, that is, $\Gamma^* = \{\, y \in V^* \mid \langle x, y\rangle \in \mathbb{Z} \text{ for all } x \in \Gamma \,\}$. It can be easily checked that Γ^* can be identified with the $\mathbb{Z}$-dual of Γ.

THEOREM 2.20 (Poisson summation formula for lattices). Let $v = \mu(V/\Gamma)$ be the volume of the lattice Γ in V. The *Poisson summation formula* is defined as

$$\sum_{x\in\Gamma} f(x) = \frac{1}{v}\sum_{y\in\Gamma^*}\widehat{f}(y).$$

Proof. We may rescale the measure by the volume, so we can assume $\mu(V/\Gamma) = 1$. By fixing a basis of Γ, we identify V with $\mathbb{R}^n$ according to the coefficients of those basis vectors, and Γ is identified with $\mathbb{Z}^n$. The space of linear maps is also identified with $\mathbb{R}^n$ and Γ^* with $\mathbb{Z}^n$, so we reduce to the usual Poisson summation for $\mathbb{R}^n$, the proof of which is given above. $\square$

2.5 Theta functions

The theta function of Jacobi makes appearances in many areas of mathematics: combinatorics, number theory, partial differential equations, and as an important link between elliptic functions and modular forms [4].

In its most general form, Jacobi's theta function is defined for $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ by

$$\Theta(z \mid \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}. \tag{2.4}$$

This function is remarkable in that it has a dual nature: when viewed as a function of z, it appears in the context of elliptic functions (since Θ is periodic with period 1 and "quasi-period" τ), but when considered as a function of τ, Θ appears in the world of modular forms, partition functions (cf. Section 3.7), and in the problem of representation of integers as sums of squares [5].

There are two significant special cases of Θ, defined by

$$\theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau}, \quad \tau \in \mathbb{H},$$
$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}, \quad t > 0.$$

The relation between these functions is given by $\theta(\tau) = \Theta(0 \mid \tau)$ and $\vartheta(t) = \theta(it)$, with $t > 0$.

We will primarily be concerned with $\theta(\tau)$, but first let us state some structural properties of Θ.

THEOREM 2.21. The function Θ satisfies the following properties:

1. Θ is entire in $z \in \mathbb{C}$ and holomorphic in $\tau \in \mathbb{H}$.

2. $\Theta(z+1 \mid \tau) = \Theta(z \mid \tau)$.

3. $\Theta(z+\tau \mid \tau) = \Theta(z \mid \tau)e^{-\pi i\tau}e^{-2\pi iz}$.

4. $\Theta(z \mid \tau) = 0$ whenever $z = 1/2+\tau/2+n+m\tau$ and $n,m \in \mathbb{Z}$.

Proof. We sketch the proof of part 1, referring the reader to [5] for the remaining steps. Suppose that $\mathrm{Im}(\tau) = t \geq t_0 > 0$ and $z = x+iy$ belongs to a bounded set in $\mathbb{C}$, say, $|z| \leq M$. Then the series defining Θ is absolutely and uniformly convergent since

$$\sum_{n=-\infty}^{\infty} \left| e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \leq C \sum_{n\geq 0} e^{-\pi n^2 t_0} e^{2\pi n M} < \infty.$$

Therefore, for each fixed $\tau \in \mathbb{H}$ the function $\Theta(\cdot \mid \tau)$ is entire, and for each fixed $z \in \mathbb{C}$, the function $\Theta(z \mid \cdot)$ is holomorphic in $\mathbb{H}$. $\square$

We will need the following theorem later when showing that the Dedekind eta function is a modular form:

THEOREM 2.22. If $\tau \in \mathbb{H}$, then

$$\Theta(z \mid -1/\tau) = \sqrt{\frac{\tau}{i}} e^{\pi i \tau z^2} \Theta(z\tau \mid \tau) \quad \text{for all } z \in \mathbb{C}. \tag{2.5}$$

Here $\sqrt{\tau/i}$ denotes the branch of the square root defined on $\mathbb{H}$, that is positive when $\tau = it,\ t > 0$.

Proof. It suffices to prove the formula for $z = x \in \mathbb{R}$ and $\tau = it$ with $t > 0$, since for each fixed $x \in \mathbb{R}$, the two sides of (2.5) are holomorphic in $\mathbb{H}$ and agree on the positive imaginary axis, and so they must be equal everywhere. Also, for a fixed $\tau \in \mathbb{H}$, both sides define holomorphic functions in z that agree on the real axis, and so they must be equal everywhere.

With $x \in \mathbb{R}$ and $\tau = it$, (2.5) becomes

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/t} e^{2\pi i n x} = t^{1/2} e^{-\pi t x^2} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} e^{-2\pi n x t}.$$

Replacing x with a, we must now prove

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2}\mathrm{e}^{-\pi n^2/t}\mathrm{e}^{-2\pi \mathrm{i} na}.$$

To do this, first observe that if $\xi \in \mathbb{R}$, then we can obtain through contour integration the following:

$$\mathrm{e}^{-\pi\xi^2} = \int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\mathrm{e}^{-2\pi \mathrm{i} x\xi}\,\mathrm{d}x.$$

This shows us that $\mathrm{e}^{-\pi x^2}$ is its own Fourier transform (cf. Section 2.4). We fix values for $t > 0$ and $a \in \mathbb{R}$, and then make a change of variables $x \mapsto t^{1/2}(x+a)$ in the above integral to get that the Fourier transform of the function

$$f(x) = \mathrm{e}^{-\pi t(x+a)^2}$$

is $\widehat{f}(\xi) = t^{-1/2}\mathrm{e}^{-\pi\xi^2/t}\mathrm{e}^{2\pi \mathrm{i} a\xi}$. Apply the Poisson summation formula to both f and $\widehat{f}$ to get the relation

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2}\mathrm{e}^{-\pi n^2/t}\mathrm{e}^{2\pi \mathrm{i} na}, \tag{2.6}$$

as desired. $\square$

We can now turn our attention to the version of the theta function defined on lattices. Let V be a $\mathbb{R}$-vector space of finite dimension endowed with a symmetric bilinear form $x.y$ which is positive and nondegenerate (that is, $x.x > 0$ if $x \neq 0$). We may identify V with V^* through this bilinear form. Let Γ be a lattice in V^*; the lattice Γ^* becomes a lattice in V (we have $y \in \Gamma^*$ if and only if $x.y \in \mathbb{Z}$ for all $x \in \Gamma$).

We will be interested in pairs (V,Γ) that satisfy the following two properties:

1. The dual Γ^* of Γ is equal to Γ.

2. For all $x \in \Gamma$, we have $x.x \equiv 0 \pmod 2$.

Let $m \geq 0$ be an integer, and denote by $r_\Gamma(m)$ the number of elements x of Γ such that $x.x = 2m$. It is easy to show that $r_\Gamma(m)$ is bounded by a polynomial in m, hence the series with integer coefficients

$$\sum_{m=0}^{\infty} r_\Gamma(m)q^m = 1 + r_\Gamma(1)q + \cdots$$

converges for $|q| < 1$, so we may define a function θ_Γ on $\mathbb{H}$ by the following:

$$\theta_\Gamma(\tau) = \sum_{m=0}^{\infty} r_\Gamma(m)q^m.$$

(Recall that $q = \mathrm{e}^{2\pi \mathrm{i} z}$.) From a simple counting argument, we have that

$$\theta_\Gamma(\tau) = \sum_{m=0}^{\infty} r_\Gamma(m)q^m = \sum_{x \in \Gamma} q^{(x.x)/2}.$$

The function θ_Γ is called the *theta function of* Γ. Since θ_Γ does converge for $|q| < 1$, it is indeed analytic and hence holomorphic on $\mathbb{H}$.

THEOREM 2.23. The function θ_Γ satisfies the equation

$$\theta_\Gamma(\mathrm{i}t) = \mathrm{i}t^{-n/2} \frac{1}{\mu(V/\Gamma)} \Theta_{\Gamma^*}\left(\frac{1}{\mathrm{i}t}\right).$$

In particular, if $\Gamma = \Gamma^* = \mathbb{Z}$, we have

$$\theta_\Gamma(\mathrm{i}t) = \frac{1}{\sqrt{t}} \theta_\Gamma\left(\frac{1}{\mathrm{i}t}\right).$$

Proof. We will apply Poisson summation to the function $f(x) = \mathrm{e}^{-\pi x.x}$, a rapidly decreasing smooth function on V. To determine the Fourier transform of f, fix an orthonormal basis for V that identifies V with $\mathbb{R}^n$ so that the measure becomes $\mathrm{d}x = \mathrm{d}x_1 \cdots \mathrm{d}x_n$ and the inner product simplifies to $f = \mathrm{e}^{-\pi(x_1^2+\cdots+x_n^2)}$. Hence the Fourier coefficient

$$\widehat{f}(x) = \int_{\mathbb{R}^n} \mathrm{e}^{-2\pi \mathrm{i}(x_1 y_1+\cdots+x_n y_n)} \mathrm{e}^{-\pi(y_1^2+\cdots+y_n^2)}\, \mathrm{d}y$$

can be realized as an iterated integral which is identical in each coordinate. Choose one such integral, complete the square in the exponent and evaluate to find the Fourier transform of $\mathrm{e}^{-\pi x^2}$ is again $\mathrm{e}^{-\pi x^2}$, and so f equals $\widehat{f}$.

The theta function has summands $\mathrm{e}^{-\pi t x \cdot x}$. Again, use the function f defined above, now for the lattice $t^{1/2}\Gamma$, which is a translation of all elements of Γ by $t^{1/2}$. Its volume in V is $t^{1/2}\mu(V/\Gamma)$ where n is the dimension of V, and its dual is $t^{-1/2}\Gamma'$ by definition. Applying the Poisson summation formula for lattices gives the desired result. $\square$

If we require the dual lattice Γ^* to be equal to Γ, we can apply Theorem 2.23 to give

$$\theta_\Gamma(-1/\mathrm{i}t) = t^{n/2}\theta_\Gamma(\mathrm{i}t).$$

Since $\theta_\Gamma(-1/z)$ and $(\mathrm{i}z)^{n/2}\theta_\Gamma(z)$ are both analytic in z, and are equal for z on the positive imaginary axis, then by analytic continuation it is true for all $z \in \mathbb{H}$. Hence we have the following:

PROPOSITION 2.24. Let Γ be a self-dual lattice. For any $z \in \mathbb{H}$,

$$\theta_\Gamma(-1/z) = (\mathrm{i}z)^{n/2}\theta_\Gamma(z).$$

THEOREM 2.25. The following statements are true:

1. The dimension n of V is divisible by 8.

2. The function θ_Γ is a modular form of weight $n/2$.

Proof. The proof of the second statement follows directly from Proposition 2.24. Using the fact that n is divisible by 8, we can rewrite the equation as

$$\theta_\Gamma(-1/z) = z^{n/2}\theta_\Gamma(z),$$

which shows that θ_Γ is indeed a modular form of weight $n/2$. □

2.5.1 Dedekind eta function

Another example of a modular form with connections to these theta functions, as well as some relevance to the study elliptic curves, is the Dedekind eta function.

DEFINITION 2.26. Let $\tau \in \mathbb{H}$. Then the *Dedekind eta function* is defined to be

$$\eta(\tau) = \mathrm{e}^{\pi \mathrm{i}\tau/12}\prod_{n=1}^{\infty}(1-\mathrm{e}^{2\pi \mathrm{i}n\tau}).$$

It is interesting to note (and somewhat surprising) that $\eta^{24}(\tau) = \Delta(z)$, the modular discriminant defined above.

It is easy to show that $\eta(\tau)$ satisfies the first (periodic) relation of a modular form of weight $1/2$; we now give the proof that it also satisfies the second relation as well:

Proof. We begin by differentiating the triple product form of the general Jacobi theta function

$$\Theta(z \mid \tau) = (1+qe^{-2\pi \mathrm{i}z})\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}\mathrm{e}^{2\pi \mathrm{i}z})(1+q^{2n+1}\mathrm{e}^{-2\pi \mathrm{i}z})$$

and evaluating it at $z_0 = 1/2+\tau/2$ to see that

$$\Theta'(z_0 \mid \tau) = 2\pi \mathrm{i}H(\tau), \qquad \text{where } H(\tau) = \prod_{n=1}^{\infty}(1-\mathrm{e}^{2\pi \mathrm{i}n\tau})^3.$$

Next, we observe that by replacing τ with $-1/\tau$ in (2.5), we obtain

$$\Theta(z \mid -1/\tau) = \sqrt{\tau/\mathrm{i}}\mathrm{e}^{\pi \mathrm{i} \tau z^2}\Theta(z\tau \mid \tau).$$

We differentiate this and evaluate it at z_0 to see that

$$\Theta'(z_0 \mid \tau) = 2\pi\mathrm{i}H(\tau) = \sqrt{\mathrm{i}/\tau}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4\tau}}\mathrm{e}^{-\frac{\pi\mathrm{i}}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{4}}\left(\frac{-2\pi\mathrm{i}}{\tau}\right)H(-1/\tau).$$

Combining these two evaluations for $\Theta'(z_0 \mid \tau)$, we have

$$\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{4}}H(\tau) = \left(\frac{\mathrm{i}}{\tau}\right)^{3/2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4\tau}}H(-1/\tau).$$

Since $\tau \in \mathbb{H}$, $\eta(\tau)$ is positive so we may take the cube root of the above to get

$$\eta(\tau) = \sqrt{\mathrm{i}/\tau}\eta(-1/\tau).$$

This identity holds for all $\tau \in \mathbb{H}$ by analytic continuation. $\square$

Vertex algebras

3.1 Notation

From here on out we assume that all vector spaces are defined over $\mathbb{C}$ and that linear transformations are $\mathbb{C}$-linear. We use $\mathrm{End}(V)$ to denote the space of all endormorphisms of a vector space V. We continue to use $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$.

We use this notation for the following formal power series:

$$V[[z,z^{-1}]] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in V \right\},$$
$$V[[z]][z^{-1}] = \left\{ \sum_{n=-M}^{\infty} a_n z^n \mid a_n \in V \right\}.$$

These form linear spaces with respect to the obvious addition and scalar multiplication.

Given a formal power series in one variable, $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$, we define its formal residue at 0 to be

$$\mathrm{Res}\, f(z)\,\mathrm{d}z = \mathrm{Res}_{z=0}\, f(z)\,\mathrm{d}z = a_{-1}.$$

3.1.1 The formal delta function

We will make use of the following important power series in two variables:

DEFINITION 3.1. The *formal delta function* [2] is

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1}.$$

This delta function can be multiplied by an arbitrary formal power series in one variable (that is, depending only on z or w) since its coefficients $a_{mn} = \delta_{m,-n-1}$ are supported on the diagonal $m+n=-1$. Carrying out such a multiplication, we obtain

$$a(w)\delta(z-w) = \sum_{n\in\mathbb{Z}} a_n w^n \sum_{m\in\mathbb{Z}} z^m w^{-m-1} = \sum_{m,n\in\mathbb{Z}} a_{m+n+1} z^m w^n,$$

so each coefficient is well-defined. This formula shows that when considered as a formal power series

$$a(z)\delta(z-w) = a(w)\delta(z-w), \tag{3.1}$$

which is the motivation for calling this the "delta function." Furthermore, from induction on (3.1) applied to $a(z) = z$, we have that

$$(z-w)^{n+1}\partial_w^n \delta(z-w) = 0. \tag{3.2}$$

3.2 Fields and locality

DEFINITION 3.2. A formal power series

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n} \in \operatorname{End} V[[z,z^{-1}]]$$

is called a *field* if for any $v \in V$ we have $a_n \cdot v = 0$ for large enough n, that is, if

$$a(z)\cdot v \in V[[z]][z^{-1}].$$

Intuitively, this means that a field is a Laurent series with coefficients in $\mathrm{End}(V)$ that truncates in the negative direction. A field defines a linear map

$$\begin{aligned} a(z) : V &\to V[[z, z^{-1}]] \\ v &\mapsto \sum_{n\in\mathbb{Z}} a_n(v) z^{-n-1}. \end{aligned}$$

Furthermore, we define the space of fields

$$\mathfrak{F}(V) = \left\{ \, a(z) \in \mathrm{End}(V)[[z, z^{-1}]] \mid a(z) \text{ is a field} \, \right\}.$$

Note that $\mathfrak{F}(V)$ is a subspace of $\mathrm{End}(V)[[z, z^{-1}]]$. It is easy to check that the product of fields is a field, and that the derivative of a field is also a field.

We call the individual endormorphisms a_n the *modes* of $a(z)$, and the elements of V the *states*. Hence V is called the *state space*.

DEFINITION 3.3. $a(z), b(z) \in \mathrm{End}(V)[[z, z^{-1}]]$ are called *mutually local* if there exists a nonnegative integer k such that

$$(z_1 - z_2)^k [a(z_1), b(z_2)] = 0. \tag{3.3}$$

Locality defines a symmetric relation which is generally neither reflexive nor transitive. Fix a *nonzero* state $\mathbf{1} \in V$. We say that $a(z) \in \mathfrak{F}(V)$ is *creative* (with respect to $\mathbf{1}$) and *creates the state u* if

$$a(z)\mathbf{1} = u + \cdots \in V[[z]].$$

We sometimes write this in the form $a(z)\mathbf{1} = u + O(z)$. In terms of modes,

$$a_n \mathbf{1} = 0, \quad n \geq 0, \quad a_{-1}\mathbf{1} = u.$$

Later, when we wish to establish the locality of the fields in the Heisenberg vertex algebra, we will avoid some tedious calculation and instead use the following general result.

LEMMA 3.4. If $a(z)$ and $b(w)$ are mutually local then $\partial_z^n a(z)$ and $\partial_w^m b(w)$ are mutually local for any $m,n \geq 0$.

Proof. We see that $\partial_z^n b(z)$ and $\partial_w^m b(w)$ are mutually local from differentiating $(z-w)^N[a(z),b(w)] = 0$, for some N, with respect to z and multiplying the result by $(z-w)$ to obtain

$$(z-w)^{N+1}[\partial_z a(z), b(w)] = 0,$$

so $\partial_z a(z)$ and $b(w)$ are mutually local. By induction, $\partial_z^n b(z)$ and $\partial_w^m b(w)$ are local for any $m,n \geq 0$. □

3.2.1 Normally ordered products

When dealing with the locality of fields, it will be useful to define a special product which essentially amounts to a lexicographic reordering of terms in the usual product. First, for notation's sake we define for $f(z) = \sum_{n\in\mathbb{Z}} f_n z^n \in \mathbb{C}((z))$,

$$f_+(z) = \sum_{n\geq 0} f_n z^n, \qquad f_-(z) = \sum_{n<0} f_n z^n.$$

DEFINITION 3.5. Let $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $b(w) = \sum_{m\in\mathbb{Z}} b_m w^{-m-1}$ be fields. The *normally ordered product* of $a(z)$ and $b(w)$ is

$$\begin{aligned} :a(z)b(w): &= a(z)_+ b(w) + b(w)a(z)_- \\ &= \sum_{n\in Z}\left(\sum_{m<0} a_m b_n z^{-m-1} + \sum_{m\geq 0} b_n a_m z^{-m-1}\right) w^{-n-1}. \end{aligned}$$

In general, the normally ordered product is neither commutative nor associative. Also, by convention, we read the normal ordering from left to right, so that

$$:a(z)b(z)c(z): = :a(z)\,(:b(z)c(z):)\,:.$$

LEMMA 3.6. The normally ordered product also satisfies the relation

$$:a(w)b(w): = \mathrm{Res}_{z=0}(\delta(z-w)_{-}a(z)b(w) + \delta(z-w)_{+}b(w)a(z)),$$

where

$$\delta(z-w)_{+} = \sum_{m\geq 0} z^m w^{-m-1}, \qquad \delta(z-w)_{-} = \sum_{m<0} z^m w^{-m-1}.$$

Proof. Since $\mathrm{Res}_{z=0}(\delta(z-w)_{\pm}a(z)) = a_{\mp}(w)$, and residue is linear, we have

$$\begin{aligned}\mathrm{Res}_{z=0}(\delta(z-w)_{-}a(z)b(w) + \delta(z-w)_{+}b(w)a(z)) &= a_{+}(w)b(w) + b(w)a_{-}(z)\\ &= :a(w)b(w):.\end{aligned}$$

□

LEMMA 3.7 (Dong's Lemma [2]). Let $a(z), b(z), c(z)$ be mutually local fields. Then $:a(z)b(z):$ and $c(z)$ are mutually local as well.

Proof. By assumption we may find r such that for all $s \geq r$,

$$(w-z)^s a(z)b(w) = (w-z)^s b(w)a(z),$$

$$(u-z)^s a(z)c(u) = (u-z)^s c(u)a(z),$$

$$(u-w)^s b(w)c(u) = (u-w)^s c(u)b(w).$$

We wish to find an integer N such that

$$(w-u)^N{:}a(w)b(w){:}c(u) = (w-u)^N c(u){:}a(w)b(w){:}.$$

Using Lemma 3.6, this will follow from the statement

$$\begin{aligned}(w-u)^N\left(\delta(z-w)_-a(z)b(w)+\delta(z-w)_+b(w)a(z)\right)c(u)\\ =(w-u)^N c(u)\left(\delta(z-w)_-a(z)b(w)+\delta(z-w)_+b(w)a(z)\right).\end{aligned} \tag{3.4}$$

By taking $N=3r$ and writing

$$(w-u)^{3r} = (w-u)^r\sum_{s=0}^{2r}\binom{2r}{s}(w-z)^s(z-u)^{2r-s},$$

we see that the terms on the left hand side of (3.4) with $r<s\le 2r$ vanish, since one factor of $(z-w)$ kills the sum $\delta(z-w)_- + \delta(z-w)_+ = \delta(z-w)$, while we will still have at least r such factors, allowing us to switch the order of $a(z), b(w)$ by their locality. The terms with $0\le s\le r$ have $(z-u)$ appearing to a power of at least r, which allows us to move $c(u)$ through $a(z)$ while also still having $(w-u)$ to the rth power, so that we can move $c(u)$ through $b(w)$. Similarly, on the right hand side, the terms with $r<s\le 2r$ will vanish, and the other terms give us the same expression as on the left hand side. This establishes (3.4) and the lemma. □

3.3 Axioms for a vertex algebra

DEFINITION 3.8. A *vertex algebra* (VA) consists of the following data:

1. (*state space*) a $\mathbb{Z}_+$-graded vector space $V=\bigoplus_{m\ge 0}V_m$,;

2. (*state-field correspondence*) a linear map

$$Y : V \to \mathfrak{F}(V),$$
$$v \mapsto Y(v,z) = \sum_{n \in Z} v_n z^{-n-1},$$

where the state $v \in V_m$ is associated to the field $Y(v,z)$ of conformal dimension m, that is, $\deg v_n = -n + m - 1$;

3. (*vacuum state*) a nonzero state $\mathbf{1} \in V$;

4. (*translation operator*) a linear operator $D \in \mathrm{End}(V)$,

which satisfy the following axioms for all $u, v \in V$:

1. (*locality axiom*) $Y(u,z) \sim Y(v,z)$, that is, all fields $Y(u,z)$ are local with respect to each other;

2. (*vacuum axiom*) $Y(|0\rangle, z) = \mathrm{Id}_V$. Furthermore, for any $v \in V$ we have $Y(v,z)|0\rangle \in V[[z]]$, so that $Y(v,z)|0\rangle$ has a well-defined value at $z = 0$, and

$$Y(v,z)|0\rangle|_{z=0} = v;$$

3. (*translation axiom*) $[D, Y(u,z)] = \partial_z Y(u,z)$ and $D|0\rangle = 0$.

It is common in the literature to refer to the state space V itself as a vertex algebra rather than $(V, Y, \mathbf{1}, D)$. Intuitively, we can think of the creativity axiom to mean that $Y(u,z)$ *creates* the state u out of the vacuum state.

3.4 The Heisenberg vertex algebra

We will now construct our first important example: the Heisenberg vertex algebra. In the context of conformal field theory, this models a single free (in the physics sense) boson. We give a concrete construction that begins by defining a particular Lie algebra and then endowing it with the structure of a vertex algebra [2].

DEFINITION 3.9. The *Heisenberg Lie algebra* $\mathfrak{h}_n$ is the $2n+1$-dimensional real Lie algebra with basis elements

$$\{\, P_1,\ldots,P_n,Q_1,\ldots,Q_n,C \,\}$$

and Lie bracket defined by

$$[P_i,P_j]=[Q_i,Q_j]=[P_i,C]=[Q_i,C]=[C,C]=0, \qquad [P_i,Q_j]=C\delta_{ij},$$

for all $i,j=1,\ldots,n$.

Now we will construct a representation of the Heisenberg Lie algebra. Let $\pi=\mathbb{C}[b_{-1},b_{-2},\ldots]$ and for $v\in\pi$, we let b_n act in the following way:

$$b_n v=\begin{cases} b_n v, & n<0 \\ n\frac{\partial}{\partial b_{-n}}v, & n\geq 0. \end{cases}$$

It follows that $[b_n,b_{-n}]=n$, or more generally, $[b_n,b_m]=n\delta_{n,-m}\mathbf{1}$. Hence, π forms a representation of the Heisenberg Lie algebra and is the state space of the Heisenberg VA. Note that π must have a $\mathbb{Z}_+$ gradation, that is, π has a basis of monomials $b_{j_1}\ldots b_{j_k}$. We assign to this monomial degree $-\sum_{i=1}^{k} j_i$, that is, we set $\deg\mathbf{1}=0$ and $\deg b_j=-j$ for all $j\leq -1$.

The operators b_n with $n<0$ are known in this context as *creation operators*, since they "create the state b_n from the vacuum $\mathbf{1}$." On the other hand, the operators b_n with $n \geq 0$ are the *annihilation operators*, since repeatedly applying them will "kill" any vector in π.

We must also fix a vacuum state $|0\rangle = \mathbf{1} \in \pi$ and also give the translation operator D, defined by the rules $D\mathbf{1} = 0$ and $[D, b_i] = -ib_{i-1}$. These formulas uniquely determine D by induction on the degree of monomials:

$$D \cdot b_k m = b_k \cdot D \cdot m + [D, b_k] \cdot m$$

for any monomial m, and so

$$D \cdot b_{j_1} \dots b_{j_k} = -\sum_{i=1}^{k} j_1 b_{j_1} \dots b_{j_1 - 1} \dots b_{j_k}.$$

We now need to define the state-field correspondence map $Y(\cdot, z)$. To the vacuum state $\mathbf{1}$, we must assign $Y(\mathbf{1}, z) = \mathrm{Id}$. The most important definition is that of the field $Y(b_{-1}, z)$, since it will generate π, and we denote $Y(b_{-1}, z)$ by $b(z)$ for convenience. We set

$$b(z) := \sum_{n \in \mathbb{Z}} b_n z^{-n-1},$$

where b_n is considered an endomorphism of π. Since $\deg b_n = -n$, $b(z)$ is indeed a field of conformal dimension one. Note that $b(z)$ is a generating function for the generators b_n of the Heisenberg Lie algebra. Next we define

$$Y(b_{-2}, z) := \partial_z b(z) = \sum_{n \in \mathbb{Z}} (-n-1) b_n z^{-n-2}.$$

By induction, we obtain

$$Y(b_{-k}, z) = \frac{1}{(k-1)!} \partial_z^{k-1} b(z).$$

We use normal ordering to define

$$Y(b_{-1}^2, z) := :b(z)^2:.$$

In general, assigning state-field correspondence maps combines the previous two cases of $b_j, j < 0$ and b_{-1}^2. We define

$$Y(b_{j_1} b_{j_2} \ldots b_{j_k}, z) := \frac{1}{(-j_1 - 1)! \cdots (-j_k - 1)!} :\partial_z^{-j_1 - 1} b(z) \cdots \partial_z^{-j_k - 1} b(z):.$$

Let us now check that π does indeed satisfy the axioms of a vertex algebra.

The statement $Y(|0\rangle, z) = \mathrm{Id}$ follows from our definition. The rest of the vacuum axiom,

$$\lim_{z \to 0} Y(v, z) |0\rangle = v, \tag{3.5}$$

follows by induction on the b_i. Start with the case $v = b_{-1}$, where

$$Y(b_{-1}, z) |0\rangle = \sum_{n \in \mathbb{Z}} b_n z^{-n-1} |0\rangle .$$

All of the non-negative b_n annihilate the vacuum, so this limit is well-defined, and has as constant coefficient b_{-1}. Next, from the above definition the vertex operator associated to each polynomial in each b_i is a normally ordered product of derivatives of the basic field $b(z)$. We only need to check that if (3.5) holds for the field

$$Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1},$$

then it holds for the field

$$Y(b_{-k}v,z) = \frac{1}{(k-1)!}{:}\partial_z^{k-1}b(z)Y(v,z){:}, \quad k > 0.$$

By definition of the normally ordered product,

$$\begin{aligned}
&\frac{1}{(k-1)!}{:}\partial_z^{k-1}b(z)Y(v,z){:} = \\
&\quad \frac{1}{(k-1)!}\sum_{m\in\mathbb{Z}}\left(\sum_{n\leq -k}(-n-1)(-n-2)\cdots(-n-k+1)b_n v_{m-n}+\right. \\
&\quad\quad \left.\sum_{n\geq 0}(-n-1)(-n-2)\cdots(-n-k+1)v_{m-n}b_n\right)z^{-k-m-1}.
\end{aligned}$$

The second sum kills $|0\rangle$, and by the inductive assumption, the first sum gives a power series with only positive powers of z, with the constant term

$$b_{-k}v_{-1}|0\rangle = b_{-k}v.$$

To check the translation axiom, first observe that we have $D|0\rangle = 0$ by construction. Next, since $[D,b_j] = -jb_{j-1}$, we have $[D,b(z)] = \partial_z b(z)$. In the same way, we can derive $[D,\partial_z^n b(z)] = \partial_z^{n+1}b(z)$. We can use the residue definition of a normal ordering from Lemma 3.6 to verify that the Leibniz rule holds for the normally product

$$\partial_z{:}a(z)b(z){:} = {:}\partial_z a(z)b(z){:} + {:}a(z)\partial_z b(z){:}.$$

This implies that if $[D,\cdot]$ acts as ∂_z on two fields, it will act like this on their normally ordered product. Through induction this implies the full translation axiom.

Finally, we need to verify that all the fields are mutually local. We begin by showing

$b(z)$ is local with itself. First we expand the bracket relation

$$[b(z), b(w)] = \sum_{n,m \in \mathbb{Z}} [b_n, b_m] z^{-n-1} w^{-m-1} = \sum_{n \in \mathbb{Z}} [b_n, b_{-n}] z^{-n-1} w^{n-1}$$
$$= \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} = \partial_w \delta(z-w).$$

From (3.2) we have that $(z-w)^2 \partial_w \delta(z-w) = 0$. This implies that $(z-w)^2 [b(z), b(w)] = 0$, and so we see from Definition 3.3 that the field $b(z)$ is local with itself. From Lemma 3.6 it follows that $\partial_z^n b(z)$ and $\partial_w^m b(w)$ are mutually local. Dong's Lemma then shows that $Y(u,z)$ and $Y(v,z)$ are mutually local for any $u, v \in \pi$.

3.5 The Virasoro vertex algebra

The Virasoro vertex algebra will become crucial to the definition of vertex operator algebras below. Consider the Lie algebra with underlying vector space with generators C and L_n for $n \in \mathbb{Z}$, and bracket relations defined as $[C, L_n] = 0$, and

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} C \qquad \text{for all } n, m \in \mathbb{Z}. \tag{3.6}$$

This is the Virasoro Lie algebra, which we denote as Vir.

Now we wish to define a family of representations of Vir. Let V be a vector space with basis given by expressions of the form $L_{i_1} L_{i_2} \cdots L_{i_n}$, with $i_1 \leq i_2 \leq \ldots \leq i_n \leq -2$, together with the vacuum vector, denoted by $\mathbf{1}$. For each $c \in \mathbb{R}$, we can define a map $f_c : \text{Vir} \to \text{End}(V)$ as follows: $f_c(C)$ acts as $c\,\text{Id}$, $f_c(L_n)$ acts according to (3.6), and we impose that for $n > 2$, we have $(f_c(L_n))v = 0$. If f_c is understood, we simply write L_n for

$f_c(L_n)$. For example,

$$L_{-2}(L_{-4}L_{-2})v = [L_{-2}, L_{-4}]L_{-2}v + L_{-4}L_{-2}L_{-2}v$$
$$= 2L_{-6}L_{-2}v + L_{-4}L_{-2}L_{-2}v,$$

and

$$L_2(L_{-4}L_{-2})v = [L_2, L_{-4}]L_{-2}v + L_{-4}L_2L_{-2}v$$
$$= 6L_{-2}L_{-2}v + L_{-4}[L_2, L_{-2}]v$$
$$= 6L_{-2}L_{-2}v + \frac{c}{2}v.$$

Observe that the action of f_c serves to put the expression into normally ordered form, where the subscripts are lexicographically ordered. We denote the representation (V, f_c) by Vir_c, and refer to c as the *central charge* of the representation.

To define the Virasoro vertex algebra, we take Vir_c as the state space with its vacuum vector $\mathbf{1}$. The gradation on Vir_c is determined by $\deg L_n = -n,\ \deg \mathbf{1} = 0$. For the translation operator, we take $D = L_{-1}$. For the vertex operators, we begin by setting

$$Y(L_{-2}\mathbf{1}, z) := T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.$$

This is the generating field of Vir_c. The expansion of the bracket relation between two generating fields is as follows [2]:

LEMMA 3.10.

$$[T(z), T(w)] = \frac{c}{12}\partial_w^3 \delta(z-w) + 2T(w)\partial_w \delta(z-w) + \partial_w T(w)\delta(z-w)$$

as a formal power series in $z^{\pm 1},\ w^{\pm 1}$.

Proof. We have

$$
\begin{aligned}
[T(z),T(w)] &= \sum_{n,m}(n-m)L_{n+m}z^{-n-2}w^{-m-2}+c\sum_{n}\frac{n^3-n}{12}z^{-n-2}w^{n-2} \\
&= \sum_{j,l}2lL_j w^{-j-2}z^{-l-1}w^{l-1}+\sum_{j,l}(-j-2)L_j w^{-j-3}z^{-l-1}w^{l} \\
&= \frac{c}{12}\sum_{l}l(l-1)(l-2)z^{-l-1}w^{l-3} \\
&= 2T(w)\partial_w\delta(z-w)+\partial_w T(w)\cdot\delta(z-w)+\frac{c}{12}\partial_w^3\delta(z-w),
\end{aligned}
$$

where we have made the substitutions $j=n+m,\ l=n+1$. □

We define the rest of the vertex operators as

$$
Y(L_{j_1}\dots L_{j_m}\mathbf{1},z)=\frac{1}{(-j_1-2)!}\cdots\frac{1}{-j_m-2)!}{:}\partial_z^{-j_1-2}D(z)\dots\partial_z^{-j_m-2}D(z){:},
$$

where $j_1\le j_2\le\cdots\le j_m\le -2$. From Lemma 3.10, we have that

$$
(z-w)^4[T(z),T(w)]=0,
$$

and so the generating field $T(z)$ is local with itself. The vacuum axiom is clearly satisfied by our choice of $\mathbf{1}$. To check the translation axiom, we must calculate the commutator of translation operator $D = L_{-1}$ and the generating field $Y(L_{-2}\mathbf{1},z)$:

$$\begin{aligned}
[L_{-1}, Y(L_{-1}\mathbf{1},z)] &= \sum_{n\in\mathbb{Z}} [L_{-1}, L_n] z^{-n-2} \\
&= \sum_{n\in\mathbb{Z}} (-n-1) L_{n-1} z^{-n-2} \\
&= \sum_{m\in\mathbb{Z}} (-m-2) L_m z^{-m-3} \\
&= \partial_z Y(L_{-1}\mathbf{1},z),
\end{aligned}$$

where we used the substitution $m = n-1$. This verifies the translation axiom.

3.6 Vertex operator algebras

A vertex algebra $(V,Y,\mathbf{1},D)$ is a *vertex operator algebra* (VOA; also *conformal vertex algebra* as in [2]) of central charge $c \in \mathbb{Z}$ if V can be decomposed as a direct sum

$$V = \bigoplus_{n\in\mathbb{Z}} V_n,$$

and there exists a non-zero *conformal vector* $\omega \in V_2$ such that the Fourier coefficients L_n of the corresponding vertex operator

$$Y(\omega,z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the defining relations of the Virasoro Lie algebra with C acting on V as $c\,\mathrm{Id}$, and in addition we have $L_{-1} = D$ and $L_0\big|_{V_n} = n\,\mathrm{Id}$.

EXAMPLE 3.11. The Virasoro vertex algebra Vir_c clearly has central charge c and conformal vector $\omega = L_{-2}\mathbf{1}$. It has the decomposition $\mathrm{Vir}_c = \bigoplus_n V_n$ where V_n is the n-eigenspace of L_0.

EXAMPLE 3.12. The Heisenberg VA π has a natural conformal vector given by

$$\omega = \frac{1}{2}b_{-1}^2$$

of central charge 1. To see that (π, ω) is indeed a VOA, we check that the Fourier coefficients of the field

$$L(z) = Y\left(\frac{1}{2}b_{-1}^2, z\right) = \frac{1}{2}{:}b(z)^2{:} = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

satisfy the Virasoro relations, that $L_{-2} = D$, and that L_0 is the degree operator.

In order to show that the field $L(z)$ satisfies the Virasoro relations, we compute $\frac{1}{2}{:}b(z)^2{:}$ and then show that the commutator $\left[\frac{1}{2}{:}b(z)^2{:}, \frac{1}{2}{:}b(w)^2{:}\right]$ satisfies the relation given in Lemma 3.10. We have

$$\begin{aligned}\frac{1}{2}{:}b(z)^2{:} &= \frac{1}{2}\left(b_+(z)b(z) + b(z)b_-(z)\right)\\ &= \frac{1}{2}\left(b_+(z)b_+(z) + 2b_+(z)b_-(z) + b_-(z)b_-(z)\right).\end{aligned}$$

Before calculating the commutator, recall that $[b_+(z), b_+(w)] = 0$ and similarly $[b_-(z), b_-(w)] = 0$. Furthermore, since $[b(z), b(w)] = \partial_w \delta(z-w)$ (which we will denote by δ), we may derive the relations

$$\delta^- := [b_-(z), b_+(w)] = \frac{1}{(z-w)^2}, \quad \text{when } |w| < |z|, \tag{3.7}$$

$$\delta^+ := [b_+(z), b_-(w)] = -\frac{1}{(w-z)^2}, \quad \text{when } |z| < |w|. \tag{3.8}$$

So we have that $\delta^- + \delta^+ = \partial_w \delta(z-w)$. Hence,

$$
\begin{aligned}
\left[\frac{1}{2}{:}b(z)^2{:}, \frac{1}{2}{:}b(w)^2{:}\right] &= \left[\frac{1}{2}b_+(z)b_+(z) + b_+(z)b_-(z) + \frac{1}{2}b_-(z)b_-(z),\right. \\
&\qquad \left.\frac{1}{2}b_+(w)b_+(w) + b_+(w)b_-(w) + \frac{1}{2}b_-(w)b_-(w)\right] \\
&= \left[\frac{1}{2}b_+(z)b_+(z), \frac{1}{2}b_+(w)b_+(w)\right] + \\
&\qquad \left[\frac{1}{2}b_+(z)b_+(z), b_+(w)b_-(w)\right] + \\
&\qquad \left[\frac{1}{2}b_+(z)b_+(z), \frac{1}{2}b_-(w)b_-(w)\right] + \\
&\qquad \left[b_+(z)b_-(z), \frac{1}{2}b_+(w)b_+(w)\right] + \\
&\qquad [b_+(z)b_-(z), b_+(w)b_-(w)] + \\
&\qquad \left[b_+(z)b_-(z), \frac{1}{2}b_-(w)b_-(w)\right] + \\
&\qquad \left[\frac{1}{2}b_-(z)b_-(z), \frac{1}{2}b_+(w)b_+(w)\right] + \\
&\qquad \left[\frac{1}{2}b_-(z)b_-(z), b_+(w)b_-(w)\right] + \\
&\qquad \left[\frac{1}{2}b_-(z)b_-(z), \frac{1}{2}b_-(w)b_-(w)\right] \\
&= b_+(w)b_+(z)\delta^+ + \frac{1}{2}\left(b_-(w)b_+(w) + b_+(z)b_-(w)\right)\delta^+ + \\
&\qquad b_+(z)b_+(w)\delta^- + b_+(w)b_-(z)\delta^+ + b_+(z)b_-(w)\delta^- + \\
&\qquad b_-(w)b_-(z)\delta^+ + \frac{1}{2}\left(b_+(w)b_-(z) + b_-(z)b_+(w)\right)\delta^- + \\
&\qquad b_-(z)b_-(w)\delta^- \\
&= b_+(z)b_+(w)\delta^+ + \frac{1}{2}b_+(z)b_-(w)\delta^+ - \frac{1}{2}\delta^+\delta^+ + \\
&\qquad \frac{1}{2}b_+(z)b_-(w)\delta^+ + b_+(z)b_+(w)\delta^- + b_+(w)b_-(z)\delta^+ + \\
&\qquad b_+(z)b_-(w)\delta^- b_-(w)b_-(z)\delta^+ + \frac{1}{2}b_+(w)b_-(z)\delta^- + \\
&\qquad \frac{1}{2}b_+(w)b_-(z)\delta^- + \frac{1}{2}\delta^-\delta^- + b_-(w)b_-(z)\delta^-.
\end{aligned}
$$

After collecting terms, we arrive at

$$\left[\frac{1}{2}{:}b(z)^2{:}, \frac{1}{2}{:}b(w)^2{:}\right] = {:}b(z)b(w){:}\partial_w \delta(z-w) + \frac{1}{2}\left(\delta^-\delta^- - \delta^+\delta^+\right). \tag{3.9}$$

From (3.7) and (3.8) we have that

$$\delta^-\delta^- = \frac{1}{(z-w)^4}, \quad \text{when } |w| < |z|,$$
$$\delta^+\delta^+ = \frac{1}{(z-w)^4}, \quad \text{when } |z| < |w|,$$

hence

$$\delta^-\delta^- - \delta^+\delta^+ = i_{|w|<|z|}\frac{1}{(z-w)^4} - i_{|z|<|w|}\frac{1}{(z-w)^4},$$

where $i_{|w|<|z|} = 1$ in the region where $|w| < |z|$, and is zero otherwise. Now observe that by taking partial derivates, we find that

$$\frac{1}{(z-w)^2} = \partial_w \frac{1}{(z-w)},$$
$$\frac{1}{(z-w)^3} = \frac{1}{2}\partial_w \frac{1}{(z-w)^2},$$
$$\frac{1}{(z-w)^4} = \frac{1}{6}\partial_w \frac{1}{(z-w)^3} = \frac{1}{6}\partial_w^3 \frac{1}{(z-w)} = \frac{1}{6}\partial_w^3 \delta(z-w),$$

thus we have

$$\frac{1}{2}\left(\delta^-\delta^- - \delta^+\delta^+\right) = \frac{1}{12}\partial_w^3 \delta(z-w).$$

Now it remains to deal with the $b(z)$ appearing in the first term of (3.9), for which we expand the normally ordered product to get

$${:}b(z)b(w){:}\partial_w \delta(z-w) = (b_+(z)b(w) + b(w)b_-(z))\partial_w \delta(z-w).$$

First, recalling (3.1), we have

$$b(w)b_{-}(z)\delta(z-w) = b(w)b_{-}(w)\delta(z-w).$$

We take the partial derivate with respect to w of both sides to get

$$\begin{aligned}\partial_w b(w)b_{-}(z)\delta(z-w) + b(w)b_{-}(z)\partial_w\delta(z-w) = \partial_w b(w)b_{-}(w)\delta(z-w)+ \\ b(w)\partial_w b_{-}(w)\delta(z-w)+ \\ b(w)b_{-}(w)\partial_w\delta(z-w)\end{aligned}$$

Again, using (3.1), we may cancel the first term of both sides to get

$$b(w)b_{-}(z)\partial_w\delta(z-w) = b(w)\partial_w b_{-}(w)\delta(z-w) + b(w)b_{-}(w)\partial_w\delta(z-w).$$

We repeat this process to see that

$$b_{+}(z)b(w)\partial_w\delta(z-w) = \partial_w b_{+}(w)b(w)\delta(z-w) + b_{+}(w)b(w)\partial_w\delta(z-w).$$

We can substitute these two new equations into (3.9) to get

$$\begin{aligned}\left[\frac{1}{2}{:}b(z)^2{:}, \frac{1}{2}{:}b(w)^2{:}\right] &= \partial_w b_{+}(w)b(w)\delta(z-w) + b_{+}(w)b(w)\partial_w\delta(z-w)+ \\ &\quad b(w)\partial_w b_{-}(w)\delta(z-w) + b(w)b_{-}(w)\partial_w\delta(z-w)+ \\ &\quad \frac{1}{6}\partial_w^3\delta(z-w) \\ &= {:}\partial_w b(w)b(w){:}\delta(z-w) + {:}b(w)b(w){:}\partial_w\delta(z-w)+ \\ &\quad \frac{1}{12}\partial_w^3\delta(z-w).\end{aligned}$$

This satisfies the relation in Lemma 3.10. The remaining two conditions are easy to verify.

We have

$$L_{-1} = \frac{1}{2} \sum_{i+j=-1} b_i b_j.$$

This operator kills the vacuum, because if $i + j = -1$ then $i \neq j$ and either $i \geq 0$ or $j \geq 0$, and thus

$$[L_{-1}, b_k] = -k b_{k-1}.$$

Therefore $L_{-1} = D$. Finally,

$$L_0 = \sum_{n>0} b_{-n} b_n = \sum_{n>0} n b_{-n} \frac{\partial}{\partial b_{-n}}.$$

Then L_0 acts as eigenvalues on the V_n, that is,

$$V_n = \{\, v \in V \mid L_0 v = nv \,\},$$

and in general L_0 acts on a monomial as

$$L_0(L_{j_1} L_{j_2} \cdots L_{j_k}) = -(j_1 + j_2 + \cdots + j_k) L_{j_1} L_{j_2} \cdots L_{j_k}.$$

3.7 Partition functions

In physics (especially statistical mechanics), the *partition function* describes the properties of an observable in a physical system. We can also view the partition function as a generating function for the expected values of random variables in our system. It is possible to calculate a partition function from a given VOA, and it is crucial that this partition function matches that of the system it is trying to model.

In general, for a VOA $(V, Y, \mathbf{1}, D)$ having state space decomposition $V = \bigoplus_{n \in \mathbb{Z}} V_n$ and central charge c, the partition function of V is

$$Z_V(q) = \operatorname{Tr} q^{L_0 - c/24} = q^{-c/24} \sum_{n\in\mathbb{Z}} \dim V_n q^n. \tag{3.10}$$

To see how this formula arises, suppose V_n has basis $v_1, \ldots, v_k$. We have $L_0 v_i = n v_i$ since L_0 acts as the eigenvalue n. Then

$$q^{L_0} v_i = \mathrm{e}^{2\pi \mathrm{i} \tau L_0} v_i = q^n v_i,$$

so we have

$$\operatorname{Tr}\big|_{V_n} q^{L_0} = \sum_{i=1}^{k} q^n = kq^n = \dim V_n q^n.$$

EXAMPLE 3.13. Consider the Heisenberg VOA with state space π and conformal vector $\omega = \frac{1}{2} b_{-1}^2$ of central charge $c = 1$. Using (3.10), we have

$$Z_\pi(q) = q^{-1/24} \sum_{n\in\mathbb{Z}} \dim \pi_n q^n = q^{-1/24} \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \frac{1}{\eta(q)},$$

where η is the Dedekind eta function discussed in Section 2.5.1, and so Z_π is in fact a modular form of weight $-1/2$. To see the second equality above, recall that the series

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$$

is a generating function, the coefficients of which count the number of monomials in one variable of each degree (in this case, there is one monomial of each degree). By taking the product of generating functions

$$\left(\frac{1}{1-t}\right)\left(\frac{1}{1-s}\right) = 1 + t + s + ts + t^2 + s^2 + \cdots,$$

and then setting $t = s$, we have a way of counting the number of monomials in two variables of each degree. So it is in this way that we can count the number of monomials q^n of each degree from the coefficients of the generating function

$$\prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + q + 2q^2 + 3q^3 + 5q^4 \cdots .$$

EXAMPLE 3.14. The partition function of the Virasoro VOA Vir_c is computed in a similar fashion to that of the Heisenberg VOA, in that we wish to find the dimension of each V_n by counting the number of monomials of weighted degree n. This differs from the calculation done for the Heisenberg VOA as there are no monomials having degree 1 in the Virasoro VOA. Hence,

$$Z_{\mathrm{Vir}_c}(q) = q^{-c/24} \sum_{n \in \mathbb{Z}} \dim V_n q^n = q^{-c/24} \prod_{n \geq 2} \frac{1}{1-q^n}.$$

Note that this partition function is *not* a modular form.

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Vita

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